

# The uses of Whitham hierarchies <sup>\*)</sup>

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We review some of the uses of Whitham hierarchies in the context of the theory of the prepotential in  $\mathcal{N} = 2$  supersymmetric gauge theories. We focus on the structure of the contact terms in the twisted topological theory, and on the connection between Whitham hierarchies and the  $u$ -plane integrals for higher rank gauge groups, trying to put together the different approaches involved in this connection. We also review two other uses of the Whitham hierarchies: the interpretation of the slow times as supersymmetry breaking parameters, and the new techniques to extract instanton corrections using the RG equations written in terms of theta functions.

## §1. Introduction

Shortly after Seiberg and Witten discovered the exact solution for the low energy effective action of  $\mathcal{N} = 2$  supersymmetric  $SU(2)$  gauge theories<sup>1)</sup>, it was realized that this solution, as well as the generalizations to other gauge groups, could be interpreted in the framework of integrable systems<sup>2),3)</sup>. This connection, which is largely based on a common description in terms of Riemann surfaces and their Jacobians, has essentially involved two ingredients: the first ingredient has been the identification of an integrable classical mechanical system whose associated spectral curve reproduces the curve describing the low energy dynamics of the gauge theory. For example, for pure Yang-Mills theories, the relevant integrable system turns out to be the periodic Toda chain<sup>4)</sup>, while in the case of the mass deformed  $\mathcal{N} = 4$  theory, the corresponding system is Calogero-Moser<sup>5)-7)</sup> (which is in turn equivalent to the Hitchin system described in<sup>3)</sup>). The second ingredient is the theory of the prepotential: once the integrable system has been identified, one considers the quasiclassical Whitham hierarchy associated to the original hierarchy, which is constructed by introducing “slow” times instead of the original, “fast” times<sup>8)</sup>. The prepotential of the effective theory turns out to be, essentially, the logarithm of the quasiclassical tau function and hence depends on the slow times of the corresponding Whitham hierarchy<sup>2),9),6)</sup>.

Although there are general and rigorous arguments showing that the effective  $\mathcal{N} = 2$  theories should be governed by integrable systems<sup>3),10)</sup>, there is for the moment no dynamical reason to explain why they are described by these particular one-dimensional mechanical systems. Nevertheless, this remarkable connection between two *a priori* unrelated fields has been very fruitful. For example, the connection to integrable systems gives a unifying approach to find the Seiberg-Witten curves for

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the different gauge groups, because both Toda and Calogero-Moser systems can be formulated for any Lie algebra. In this paper, we will focus on the second aspect of the connection, namely the theory of the prepotential in the framework of Whitham hierarchies, and we will try to show that this framework is not only an elegant way to describe the prepotential, but also the appropriate point of view to understand some important aspects of  $\mathcal{N} = 2$  gauge theories. The main reason for this is that within the Whitham approach one can consider an “enlarged” prepotential with extra parameters. Consider, for simplicity, the case of pure  $SU(N)$  super Yang-Mills theory. In this case, the usual Seiberg-Witten prepotential  $\mathcal{F}(a^I, \Lambda)$  is a holomorphic function of the coordinates of the moduli space,  $a^I$ , where  $I = 1, \dots, N-1$ , and the dynamically generated scale of the theory,  $\Lambda$ . The theory of Whitham hierarchies provides a generalized prepotential  $\mathcal{F}(a^I, T_n)$ , where  $T_n$ ,  $n = 1, \dots, N-1$ , are the slow times of the Whitham hierarchy. This prepotential is a *deformation* of the Seiberg-Witten prepotential, in the sense that, if we put  $T_1 = \Lambda$ ,  $T_{n>1} = 0$ , we recover the original function  $\mathcal{F}(a^I, \Lambda)$ .

There are two contexts where deformations of the prepotential can be relevant. The first context is the study of contact terms in the twisted version of  $\mathcal{N} = 2$  gauge theories. These contact terms, as we will explain, appear when one computes the generating functional of a certain class of operators, and the source terms for insertions of these operators can be regarded as deformations of the original action. The other context is the study of soft supersymmetry breaking terms, which can also be understood as deformations of the theory. In both cases, the deformations of the action can be described in terms of the Whitham time variables of the enlarged prepotential. In this paper, we will mainly focus on the relation between the contact terms in twisted  $\mathcal{N} = 2$  theories and the theory of the prepotential from Whitham hierarchies. The issue of contact terms has been previously addressed in <sup>11)-17)</sup> from different points of view. We will try to put together these approaches using the relation to integrable systems as a unifying framework. We will also review the uses of Whitham hierarchies in the study of soft supersymmetry breaking, as explained in <sup>18)</sup>.

An extra bonus of the Whitham approach to the theory of the prepotential is a set of renormalization group (RG) equations for the prepotential in terms of theta functions associated to the root lattice of the gauge group <sup>17)</sup>. It was shown in <sup>18)</sup> that these equations give an elegant way to derive the instanton expansion of the prepotential following a recursive procedure. We will also review very briefly the strategy of the computation, and we will present some of the results obtained so far.

The organization of the paper is as follows: in section 2, we explain the structure of contact terms in twisted  $\mathcal{N} = 2$  Yang-Mills theory. In section 3, we show how the contact terms can be understood from the point of view of Whitham hierarchies. In section 4, we study the connection between the blowup formula in twisted Yang-Mills theory and the Toda-Whitham approach. In section 5, we review, following <sup>18)</sup>, the uses of Whitham hierarchies in the problem of soft supersymmetry breaking in  $\mathcal{N} = 2$  theories. In section 6 we present some results on instanton expansions using the RG equations. Finally, in section 7 we state some conclusions and open problems.

## §2. Contact terms in twisted $\mathcal{N}=2$ Yang-Mills theory

### 2.1. Twisted $\mathcal{N}=2$ Yang-Mills theory

One of the most important aspects of  $\mathcal{N}=2$  Yang-Mills theories is their relation to Donaldson theory. It is a well-known fact that any  $\mathcal{N}=2$  theory in four dimensions can be “twisted” to obtain a topological quantum field theory (TQFT), *i.e.* a quantum field theory whose correlation functions are formally metric-independent (for a review of TQFT and the twisting procedure, see for example<sup>19), 20).</sup>) In the case of  $\mathcal{N}=2$  Yang-Mills, the correlation functions of the twisted theory are in fact the famous Donaldson invariants of four-manifolds<sup>21).</sup> In this section, we will consider some aspects of this twisted theory for the gauge group  $SU(N)$  on a simply-connected four-manifold  $X$ . In the following,  $r = N - 1$  will denote the rank of the group.

The first thing to do is to identify the gauge-invariant operators of the twisted theory. The twisted theory is characterized by the existence of a BRST charge or topological charge  $Q$  (which is in fact a particular combination of the original supersymmetric charges of the  $\mathcal{N}=2$  theory), and the operators of the twisted theory have to be BRST invariant. These operators are called *observables*. The simplest observables of the theory are precisely the Casimirs of the gauge group, which we will take to be the elementary symmetric polynomials in the eigenvalues of the complex scalar field  $\phi$  of the  $\mathcal{N}=2$  vector multiplet:

$$\mathcal{O}_k = \frac{1}{k} \text{Tr} \phi^k + \text{lower order terms}, \quad k = 2, \dots, N. \quad (2.1)$$

Starting with these operators, one can generate the rest of the observables through the so-called *descent procedure*. To do this, one needs another operator in the twisted theory, the descent operator  $G_\mu$  which has spin one and comes from another combination of the SUSY charges. Acting with  $G_\mu$   $n$  times on the Casimir operators  $\mathcal{O}_k$  one generates an  $n$ -form, The integration of this  $n$ -form on an  $n$ -cycle on  $X$  is another observable, called the  $n$ -th topological descendant of  $\mathcal{O}_k$ . As we are assuming that the manifold  $X$  is simply connected, the topological descendants will be constructed with two-cycles  $S$  in  $X$ :

$$I_k(S) = \int_S G^2 \mathcal{O}_k = \frac{1}{k} \int_S \text{Tr}(\phi^{k-1} F) + \dots, \quad (2.2)$$

where  $F$  is the Yang-Mills field strength. The basic problem of the TQFT is to compute the generating function for the correlators of observables:

$$Z(p_k, f_k, S) = \langle \exp \left( \sum_k (p_k \mathcal{O}_k + f_k I_k(S)) \right) \rangle_{\text{twisted theory}}. \quad (2.3)$$

Notice that, in general,  $S$  will be an arbitrary linear combination of basic two-cycles  $S_i$ ,  $i = 1, \dots, b_2(X)$ , *i.e.*  $S = \sum_{i=1}^{b_2(X)} t_i S_i$ , therefore

$$I_k(S) = \sum_{i=1}^{b_2(X)} t_i I_k(S_i). \quad (2.4)$$

In total, we have  $r \cdot b_2(X)$  independent operators  $I_k(S_i)$ .

The computation of (2.3) can be done using the low energy exact solution of  $\mathcal{N} = 2$ ,  $SU(N)$  Yang-Mills theory. This solution is encoded in the hyperelliptic curve<sup>22)</sup>

$$y^2 = P^2(x, u_k) - 4\Lambda^{2N}, \quad (2.5)$$

where  $P(x, u_k) = x^N - \sum_{k=2}^N u_k x^{N-k}$  is the characteristic polynomial of  $SU(N)$ , and  $u_k = \langle \mathcal{O}_k \rangle$  are the VEVs of the Casimir operators (2.1). This curve has genus  $g = r$ , and, as explained in<sup>2), 4)</sup>, it can be identified with the spectral curve of the  $N$  site periodic Toda lattice. Associated to this curve, there is a meromorphic differential of the second kind, with a double pole at infinity, that can be explicitly written as:

$$dS_{SW} = P'(x, u_k) \frac{xdx}{y}. \quad (2.6)$$

This one-form satisfies the equation:

$$\frac{\partial dS_{SW}}{\partial u_{k+1}} = \omega_k, \quad (2.7)$$

where  $\omega_k = x^{k-1}dx/y$  is a basis of holomorphic differentials. Let  $\gamma^I$  and  $\gamma_{D,I}$  denote a symplectic basis of homology cycles for this curve,  $I = 1, \dots, r$ . The  $a^I$  variables of the prepotential  $\mathcal{F}(a^I, \Lambda)$ , for the duality frame associated to the cycles  $\gamma^I$ , are given by the integrals over these cycles of  $dS_{SW}$

$$a^I(u_k, \Lambda) = \frac{1}{2\pi i} \oint_{\gamma^I} \frac{xP'(u_k, x)}{\sqrt{P^2(u_k) - 4\Lambda^{2N}}} dx. \quad (2.8)$$

The same expression holds for the dual variables  $a_{D,I}$  with  $\gamma_{D,I}$  instead of  $\gamma^I$ . Finally, the effective gauge couplings  $\tau_{IJ}$  are just the components of the Riemann matrix associated to the hyperelliptic curve (2.5). A fundamental aspect of the low-energy description of  $\mathcal{N} = 2$  gauge theories is that one has to choose a duality frame, which in the language of Riemann surfaces can be understood as a choice of the symplectic basis  $\gamma^I, \gamma_{D,I}$ . Different duality frames are related by symplectic transformations of the form

$$\Gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2r, \mathbf{Z}). \quad (2.9)$$

## 2.2. The integral over the Coulomb branch

The moduli space of the hyperelliptic curve (2.5) is parametrized by the VEVs of the Casimirs  $u_k$ . For some values of these VEVs, the curve will be singular. The singular locus is precisely the divisor  $\mathcal{D}$  defined by the vanishing locus of the discriminant of (2.5),  $\Delta_\Lambda = 0$ . It is well known that on this divisor there are extra BPS states that become massless. The Coulomb branch of the  $\mathcal{N} = 2$  theory is then defined as

$$\mathcal{M}_{\text{Coulomb}} = \mathbf{C}^r - \mathcal{D}. \quad (2.10)$$

The basic principle to compute the generating function (2.3) has been introduced by Moore and Witten in<sup>11)</sup> and states that this function is given by the sum of two

different contributions: one comes from the Coulomb branch, and the other comes from the divisor  $\mathcal{D}$ :

$$Z = Z_{\text{Coulomb}} + Z_{\mathcal{D}}. \quad (2.11)$$

The first piece is given by an integral over the Coulomb branch, while the second piece involves, in general, Seiberg-Witten invariants associated to the extra massless BPS states. In this paper, we will focus on  $Z_{\text{Coulomb}}$  following<sup>(12), (13)</sup>. As shown in<sup>(11), (13)</sup>, this contribution is different from zero only when  $b_2^+(X) = 1$ . The explicit expression of this integral can be worked out using the low-energy effective action encoded in (2.5) and reads

$$Z_{\text{Coulomb}} = \int_{\mathcal{M}_{\text{Coulomb}}} [dad\bar{a}] A(u_k)^\chi B(u_k)^\sigma e^{\sum p_k u_k + S^2 \sum f_k f_l T_{k,l}} \Psi. \quad (2.12)$$

The integrand of (2.12) has various ingredients. First of all, there is a gravitational part first studied in<sup>(23)</sup> in the  $SU(2)$  case, and generalized in<sup>(12), (13), (15)</sup> to simply-laced groups. This part involves the factors:

$$\begin{aligned} A(u_k)^\chi &= \alpha^\chi \left( \det \frac{\partial u_k}{\partial a^I} \right)^{\chi/2}, \\ B(u_k)^\sigma &= \beta^\sigma \Delta_A^{\sigma/8}. \end{aligned} \quad (2.13)$$

The first factor is a modular form of weight  $(-\chi/2, 0)$ , and  $B(u_k)$  is a modular invariant. In these equations,  $\alpha, \beta$  are constants. Notice that the quantities involved here are very natural quantities associated to the hyperelliptic curve (2.5), namely, the determinant of the matrix of periods of  $\omega_k$ , and the discriminant of the curve.

Another ingredient in (2.12) is  $\Psi$ , which is given by a sum over a lattice  $\Gamma$ . In order to write this quantity, let's introduce the following notation:

$$V_I \equiv \sum_k f_k \frac{\partial u_k}{\partial a^I}. \quad (2.14)$$

We also need some geometrical ingredients. As we are on a manifold of  $b_2^+(X) = 1$ , given a metric  $g$  there exists a unique anti-self-dual form  $\omega \in H^{2,+}(X, \mathbf{R})$  such that  $\omega^2 = 1$ . The self-dual part of a two-form is then given by  $\lambda_+ = (\lambda, \omega)$ , where  $(\cdot, \cdot)$  is the usual product in cohomology. The lattice sum in  $\Psi$  comes essentially from the evaluation of the partition function of the photons in the effective  $U(1)^r$  gauge theory. A topological sector in the effective theory is specified by  $r$  2-forms  $\lambda^I \in H^2(X, \mathbf{R})$ . The lattice  $\Gamma$  consists of vectors of the form

$$\vec{\lambda} = \sum_{I=1}^r \lambda^I \vec{\alpha}_I, \quad (2.15)$$

where  $\{\vec{\alpha}_I\}_{I=1, \dots, r}$  are the simple roots of  $SU(N)$ .\*) In terms of these quantities,  $\Psi$  can be written as follows:

$$\Psi = (\det \text{Im} \tau)^{-1/2} \exp \left[ \frac{1}{8\pi} V_J [(\text{Im} \tau)^{-1}]^{JK} V_K S_+^2 \right]$$

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\*) For simplicity, we are assuming that there are no magnetic fluxes turned on. The general case is analyzed in<sup>(13)</sup>.

$$\begin{aligned}
& \times \sum_{\vec{\lambda} \in \Gamma} \exp \left[ -i\pi \bar{\tau}_{IJ}(\lambda_+^I, \lambda_+^J) - i\pi \tau_{IJ}(\lambda_-^I, \lambda_-^J) - i\pi(\vec{\lambda} \cdot \vec{\rho}, w_2(X)) - iV_I(S, \lambda_-^I) \right] \\
& \times \int \prod_{I=1}^r d\eta^I d\chi^I \exp \left\{ -\frac{i\sqrt{2}}{16\pi} \bar{\mathcal{F}}_{IJK} \eta^I \chi^J [4\pi(\lambda_+^K, \omega) + i(\text{Im}\tau)^{KL} V_L(S, \omega)] \right. \\
& \quad \left. + \frac{1}{64\pi} \bar{\mathcal{F}}_{KLI} (\text{Im}\tau)^{IJ} \bar{\mathcal{F}}_{JPQ} \eta^K \chi^L \eta^P \chi^Q \right\}, \tag{2.16}
\end{aligned}$$

where  $\vec{\rho}$  is the Weyl vector,  $w_2(X)$  is the second Stiefel-Whitney class of  $X$ , and  $\eta^I, \chi^I$  are Grassmannian coordinates which arise from the zero modes of the fermion fields in the theory.  $\mathcal{F}_{IJK}$  denote the third derivatives of the prepotential. Notice that  $\Psi$  depends explicitly on the metric of  $X$  through the two-form  $\omega$ . It is precisely this dependence which leads to the wall-crossing phenomena of Donaldson theory in manifolds with  $b_2^+(X) = 1$ , when the gauge group is  $SU(2)$ <sup>11</sup>.

In (2.12) we have also included some terms of the form  $T_{k,l} S^2$  in the exponential. These terms, which are proportional to the intersection form of the two-cycles, are the *contact terms* that we want to study. To understand the origin of these terms, notice that the operators  $I_k(S)$  that we have introduced in (2.4) are non-local operators. These operators have a low-energy counterpart  $\tilde{I}_k(S)$  which can be obtained using the descent procedure in the effective abelian theory, and have been included in  $\Psi$ . But if we consider products of operators, and we go to the low-energy theory, we rather expect an extra contribution localised on the intersection of the surfaces that support the operators:

$$I_k(S_1) I_k(S_2) \rightarrow \tilde{I}_k(S_1) \tilde{I}_k(S_2) + T_{k,l}(S_1 \cap S_2). \tag{2.17}$$

This is precisely what we have taken into account in (2.12) by introducing the contact terms  $T_{k,l}$ . It is important to notice that these terms are not predicted *a priori* by the Seiberg-Witten solution for the low-energy effective action. One has to find extra conditions in order to be able to find their structure.

### 2.3. The structure of the contact terms

To have more information on the contact terms, the first step is to write the integrand of (2.12) in a way that makes manifest the behavior under symplectic transformations. One introduces, then, the following generalized Siegel-Narain theta function:

$$\begin{aligned}
\Theta_\Gamma(\tau_{IJ}, \alpha_I, \beta^I; P, \xi_I) &= \exp \left[ -i\pi(\alpha_I, \beta^I) + \frac{\pi}{2} \left( \xi_{I,+} (\text{Im}\tau)^{IJ} \xi_{J,+} - \xi_{I,-} (\text{Im}\tau)^{IJ} \xi_{J,-} \right) \right] \\
&\times \sum_{\vec{\lambda} \in \Gamma} \exp \left[ -i\pi \bar{\tau}_{IJ}(\hat{\lambda}_+^I, \hat{\lambda}_+^J) - i\pi \tau_{IJ}(\hat{\lambda}_-^I, \hat{\lambda}_-^J) - 2\pi i(\hat{\lambda}^I, \xi_I) + 2\pi i(\hat{\lambda}^I, \alpha_I) \right], \tag{2.18}
\end{aligned}$$

where  $\hat{\lambda}^I = \lambda^I + \beta^I$ . If we take

$$\xi_I \equiv \frac{1}{2\pi} V_I S_- + \frac{\sqrt{2}}{16\pi} \bar{\mathcal{F}}_{IJK} \eta^J \chi^K \omega,$$

$$\beta^I = 0, \quad \alpha_I = \frac{1}{2}w_2(X), \quad I = 1, \dots, r, \quad (2.19)$$

the lattice sum  $\Psi$  can be written as

$$\begin{aligned} \Psi &= \exp\left[\frac{S^2}{8\pi}V_I[(\text{Im}\tau)^{-1}]^{IJ}V_J\right](\det \text{Im}\tau)^{-1/2} \\ &\times \int \prod_{I=1}^r d\eta^I d\chi^I \exp\left[\frac{\sqrt{2}}{16\pi}\mathcal{F}_{IJK}\eta^I\chi^J(\text{Im}\tau)^{KL}V_L(S, \omega)\right]\Theta_\Gamma(\tau_{IJ}, \alpha_I, \beta^I; P, \xi_I). \end{aligned} \quad (2.20)$$

It is easy to check<sup>13)</sup> that  $\widehat{\Psi} = \exp\left[-\frac{S^2}{8\pi}V_I[(\text{Im}\tau)^{-1}]^{IJ}V_J\right]\Psi$  is a modular form of weights  $(b_2(X)/2, -1)$  (one has to take into account that  $b_2^+(X) = 1$ ). The Coulomb integral then reads:

$$\begin{aligned} Z_{\text{Coulomb}} &= \int_{\mathcal{M}_{\text{Coulomb}}} [dud\bar{u}] \exp\left[\left(\sum f_k f_l T_{k,l} + \frac{1}{8\pi}V_I[(\text{Im}\tau)^{-1}]^{IJ}V_J\right)S^2\right] \\ &\times \left|\det\left(\frac{\partial a^I}{\partial u_k}\right)\right|^2 A(u_k)^\chi B(u_k)^\sigma \widehat{\Psi}. \end{aligned} \quad (2.21)$$

The factor in the second line of (2.21) is a modular invariant (notice that, as  $X$  is simply connected,  $\chi = 2 + b_2(X)$ ). We then see that this expression for the generating function (except for the exponential involving  $S^2$ ) is the integral of a duality invariant object over a moduli space parametrized by the VEVs of the Casimirs, which are duality invariant coordinates. This is precisely what we expect on physical grounds: the generating function is a physical quantity and cannot depend on the choice of duality frame in the effective theory. This argument in fact forces the exponent in  $S^2$  to be a modular invariant as well, and this gives the first constraint on the contact terms: the quantity

$$T_{k,l} + \frac{1}{8\pi} \frac{\partial u_k}{\partial a^I} [(\text{Im}\tau)^{-1}]^{IJ} \frac{\partial u_l}{\partial a^J} \quad (2.22)$$

must be invariant under the action of the symplectic group  $\text{Sp}(2r, \mathbf{Z})$ . The other constraint on the contact terms has to do with their physical origin: these terms are quantum corrections and vanish at tree level, therefore they have to go to zero in the semiclassical region of moduli space (*i.e.* when  $\Lambda/a^I \rightarrow 0$ ). The duality transformation of the second term in (2.22) is easily worked out, and one finds that under a duality transformation it is shifted by:

$$-\frac{i}{4\pi} \frac{\partial u_k}{\partial a^I} [(C\tau + D)^{-1}C]^{IJ} \frac{\partial u_l}{\partial a^J}. \quad (2.23)$$

The transformation of the contact term should compensate for this shift. Summarizing the discussion so far, we have found the following constraints for the contact terms:

- They transform under an element of  $\text{Sp}(2r, \mathbf{Z})$  as follows:

$$T_{k,l} \rightarrow T_{k,l} + \frac{i}{4\pi} \frac{\partial u_k}{\partial a^I} [(C\tau + D)^{-1}C]^{IJ} \frac{\partial u_l}{\partial a^J}. \quad (2.24)$$

- $T_{k,l} \rightarrow 0$  semiclassically.

As shown in <sup>11)</sup>, these two properties determine the contact terms unambiguously. The problem, of course, is to find explicit expressions for them.

In the case of the pure Yang-Mills theory with gauge group  $SU(2)$ , the only contact term is  $T_{2,2}$ . The solution to the constraints above was found in <sup>11)</sup>, and the result was expressed in terms of the Eisenstein series  $E_2(\tau)$ . A general procedure to find  $T_{2,2}$  was described in <sup>12), 13)</sup> by using the so-called RG equation of  $\mathcal{N} = 2$  theories <sup>27), 28)</sup>, as well as some aspects of the formalism developed in <sup>24)–26)</sup>. The RG equation for  $\mathcal{N} = 2$  theories states that <sup>27), 28)</sup>

$$\frac{\partial \mathcal{F}}{\partial \tau_0} = \frac{1}{4} u_2, \quad (2.25)$$

where  $\tau_0$  is defined as follows: for asymptotically free theories,  $\Lambda^\beta = e^{\pi i \tau_0}$ , where  $\beta$  is the coefficient of the one-loop beta function (for example,  $\beta = 2N$  for  $SU(N)$ ). For the self-dual theories,  $\tau_0$  is the microscopic coupling constant. The key point is to consider now the second derivatives of the prepotential. On the one hand, from (2.25) one has that

$$\frac{\partial^2 \mathcal{F}}{\partial a^I \partial \tau_0} = \frac{1}{4} \frac{\partial u_2}{\partial a^I}. \quad (2.26)$$

On the other hand, the fact that  $\tau_0$  is invariant under duality transformations implies the following transformation law <sup>25)</sup>:

$$\frac{\partial^2 \mathcal{F}}{\partial \tau_0^2} \rightarrow \frac{\partial^2 \mathcal{F}}{\partial \tau_0^2} - \frac{\partial^2 \mathcal{F}}{\partial \tau_0 \partial a^I} [(C\tau + D)^{-1}]^I{}_J C^{JK} \frac{\partial^2 \mathcal{F}}{\partial \tau_0 \partial a^K}. \quad (2.27)$$

This is precisely the shift we found in (2.24) under symplectic transformations. It is easy to see that  $\partial^2 \mathcal{F} / \partial \tau_0^2$  is zero semiclassically (this is related to the fact that the classical prepotential is linear in the gauge coupling constant). It follows that the contact term associated to the quadratic Casimir can be written as

$$T_{2,2} = \frac{4}{\pi i} \frac{\partial^2 \mathcal{F}}{\partial \tau_0^2}. \quad (2.28)$$

$T_{2,2}$  can be in fact evaluated very explicitly in the  $SU(2)$  case <sup>12), 26)</sup>. For the theories with  $N_f \leq 3$ , one finds:

$$T_{2,2} = -\frac{1}{24} E_2(\tau) \left( \frac{du}{da} \right)^2 + \frac{1}{3} \left( u + \delta_{N_f,3} \frac{\Lambda_3^2}{64} \right), \quad (2.29)$$

and for the  $N_f = 4$  case one has:

$$T_{2,2} = -\frac{1}{24} E_2(\tau) \left( \frac{du}{da} \right)^2 + E_2(\tau_0) \frac{u}{3} + \frac{1}{9} R E_4(\tau_0), \quad (2.30)$$

where  $R = \sum_{a=1}^4 m_a^2 / 2$ , and  $E_2, E_4$  are the normalized Eisenstein series. This provides an elegant solution to our problem for this particular contact term. The authors of <sup>15)</sup> found equivalent expressions for the  $SU(2)$  theories in the massless case and for  $N_f \leq 3$ .



The approach based on the RG equation suggests how to find  $T_{k,l}$  for any  $k, l$ . If one is able to find some additional variables  $T_n$ ,  $n = 2, \dots, N-1$  in the prepotential, which are invariant under duality and such that

$$\frac{\partial \mathcal{F}}{\partial T_n} \sim u_{n+1}, \quad (2.31)$$

then  $\partial^2 \mathcal{F} / \partial T_k \partial T_l$  will have, essentially, the duality properties of the contact term  $T_{k+1,l+1}$ , and one is halfway to the solution of the problem. The two obvious questions are: Can we find an explicit construction of the generalized variables  $T_n$ ?, and: Can we compute the second derivatives  $\partial^2 \mathcal{F} / \partial T_k \partial T_l$  in terms of elementary data associated to the Seiberg-Witten solution? These two questions are answered in <sup>17)</sup> in the affirmative through the use of the Whitham hierarchy approach to the prepotential, which will be the subject of the next section.

### §3. Whitham hierarchies and contact terms

#### 3.1. Whitham hierarchy and Seiberg-Witten solution

The approach to the theory of the prepotential based on Whitham hierarchies has been developed in <sup>8), 9), 6)</sup>. In <sup>17)</sup> it was shown that, using this approach, one can derive RG equations with the structure (2.31). We will follow here <sup>17)</sup> and also <sup>18)</sup>.

The usual Seiberg-Witten differential is a meromorphic differential with a second-order pole at infinity, and such that its variations with respect to the moduli  $u_k$  are holomorphic differentials. In order to deform the Seiberg-Witten theory and to embed it in a larger framework, one considers a series of meromorphic differentials  $d\hat{\Omega}_n$  (in the notation of <sup>17)</sup>), with poles of order  $n+1$  at infinity and satisfying the condition

$$\frac{\partial d\hat{\Omega}_n}{\partial \text{moduli}} = \text{holomorphic}. \quad (3.1)$$

One then introduces a generating functional for these one-forms with auxiliary parameters  $T_n$ ,  $n \geq 1$ :

$$dS = \sum_{n \geq 1} T_n d\hat{\Omega}_n. \quad (3.2)$$

The parameters  $T_n$  are precisely the slow times of the Whitham hierarchy. One of the results of <sup>17)</sup> is an explicit expression for the meromorphic one-forms  $d\hat{\Omega}_n$ :

$$d\hat{\Omega}_n = P_+^{n/N}(x) \frac{P'(x)dx}{\sqrt{P^2 - 4}}, \quad (3.3)$$

where  $\left(\sum_{k=-\infty}^{\infty} c_k \lambda^k\right)_+ = \sum_{k=0}^{\infty} c_k \lambda^k$ . It is easy to check that  $P_+^{1/N}(x) = x$ , therefore  $dS(T_1, T_{n \geq 2} = 0) = T_1 dS_{SW}(\Lambda = 1)$ . We then see that we recover the usual Seiberg-Witten differential when  $T_{n \geq 2} = 0$ . Starting with the enlarged differential (3.2), one can construct a deformation of the usual Seiberg-Witten theory. For instance, one defines the periods of  $dS$  as

$$\alpha^I(u_k, T_1, T_2, \dots) = \sum_{n \geq 1} \frac{T_n}{2\pi i} \oint_{\gamma^I} d\hat{\Omega}_n$$

$$\begin{aligned}
&= \sum_{n \geq 1} \frac{T_n}{2\pi i} \oint_{\gamma^I} \frac{P(u_k)_+^{n/N} P'(u_k)}{\sqrt{P^2(u_k) - 4}} d\lambda \\
&= T_1 a^I(u_k, \Lambda = 1) + \mathcal{O}(T_{n>1}), \tag{3.4}
\end{aligned}$$

where  $a^I(u_k, \Lambda)$  is the usual Seiberg-Witten period of  $dS_{SW}$ . In the same way, one defines the  $\alpha_{D,I}$  with the expression (3.4) but with the  $\gamma_{D,I}$  replacing the  $\gamma^I$ . Following now the usual steps in rigid special geometry, we obtain an enlarged prepotential  $\mathcal{F}(\alpha^I, T_n)$ , taking  $\alpha^I$  and the slow times as independent variables.

We will now make more precise the relation with the usual Seiberg-Witten solution, following<sup>18)</sup>. We define the rescaled times  $\hat{T}_n$  and VEVs  $\hat{u}_k$  as follows:

$$\hat{T}_n = T_n/T_1^n, \quad \hat{u}_k = T_1^k u_k, \tag{3.5}$$

with  $\hat{T}_1 = 1$ . It is easy to see that (3.4) can be written as

$$\alpha^I(u_k, T_1, \hat{T}_2, \hat{T}_3, \dots) = \sum_{n \geq 1} \frac{\hat{T}_n}{2\pi i} \oint_{\gamma^I} \frac{P_+^{n/N}(\hat{u}_k) P'(\hat{u}_k)}{\sqrt{P^2(\hat{u}_k) - 4T_1^{2N}}} d\lambda. \tag{3.6}$$

In particular, after setting  $\hat{T}_2 = \hat{T}_3 = \dots = 0$  we find that

$$\alpha^I(u_k, T_1, \hat{T}_{n>1} = 0) = T_1 a^I(u_k, 1) = a^I(\hat{u}_k, \Lambda = T_1).$$

In conclusion, we may identify  $T_1$  with  $\Lambda$  in the submanifold  $\hat{T}_2 = \hat{T}_3 = \dots = 0$ , provided the moduli space is parametrized with  $\hat{u}_k$  instead of  $u_k$ . This shows that the Whitham hierarchy approach to the prepotential gives a deformation of the usual Seiberg-Witten theory, in such a way that the quantum scale can be identified with the first slow time of the Whitham hierarchy.

### 3.2. Derivatives of the prepotential and contact terms

As we noticed in the last section, the slow times would be useful for the problem of the contact terms if they were “dual” to the higher Casimirs, in the sense of (2.31). We should then compute the derivatives of the prepotential  $\mathcal{F}(\alpha, T)$ . This computation was the main result of<sup>17)</sup>. One of the basic ingredients in the answer is the Riemann theta function  $\Theta[\vec{\alpha}, \vec{\beta}](\vec{\xi}|\tau)$  and characteristics  $\vec{\alpha} = (\alpha_1, \dots, \alpha_r)$ ,  $\vec{\beta} = (\beta_1, \dots, \beta_r)$ :

$$\Theta[\vec{\alpha}, \vec{\beta}](\vec{\xi}|\tau) = \sum_{n_K \in \mathbf{Z}} \exp \left[ i\pi \tau_{IJ} (n_I + \beta_I)(n_J + \beta_J) + 2\pi i (n_I + \beta_I)(\xi_I + \alpha_I) \right]. \tag{3.7}$$

The derivatives of the prepotential turn out to be:

$$\begin{aligned}
\frac{\partial \mathcal{F}}{\partial T_n} &= \frac{\beta}{2\pi i n} \sum_m m T_m \mathcal{H}_{m+1, n+1}, \\
\frac{\partial^2 \mathcal{F}}{\partial \alpha^I \partial T^n} &= \frac{\beta}{2\pi i n} \frac{\partial \mathcal{H}_{n+1}}{\partial a^I}, \\
\frac{\partial^2 \mathcal{F}}{\partial T_m \partial T_n} &= -\frac{\beta}{2\pi i} \left( \mathcal{H}_{m+1, n+1} + \frac{\beta}{mn} \frac{\partial \mathcal{H}_{m+1}}{\partial a^I} \frac{\partial \mathcal{H}_{n+1}}{\partial a^J} \frac{1}{i\pi} \partial_{\tau_{IJ}} \log \Theta_E(0|\tau) \right). \tag{3.8}
\end{aligned}$$

In these equations,  $\Theta_E(0|\tau)$  denotes Riemann's theta function with a certain characteristic  $E$ , evaluated at the origin;  $\beta = 2N$ ,  $m, n = 1, \dots, r = N - 1$ , and derivatives with respect to  $T_n$  are taken at constant  $\alpha^I$ . According to<sup>17)</sup>, the characteristic  $E$  appearing in (3.10)–(3.11) is an even, half-integer characteristic associated to a particular splitting of the roots of the discriminant. Notice that the characteristic  $E$  depends on the duality frame. An explicit expression for  $E$  in the electric frame will be given in the next section. The functions  $\mathcal{H}_{m,n}$  are certain homogeneous combinations of the Casimirs  $u_k$ , given by

$$\mathcal{H}_{m+1,n+1} = \frac{N}{mn} \text{res}_\infty \left( P^{m/N}(x) dP_+^{n/N}(x) \right) = \mathcal{H}_{n+1,m+1} ,$$

and

$$\mathcal{H}_{m+1} \equiv \mathcal{H}_{m+1,2} = \frac{N}{m} \text{res}_\infty P^{m/N}(x) dx = u_{m+1} + \mathcal{O}(u_m).$$

Here  $\text{res}_P$  stands for the usual Cauchy residue at the point  $P$ . We have for instance  $\mathcal{H}_{2,2} = \mathcal{H}_2 = u_2$ ,  $\mathcal{H}_{3,2} = \mathcal{H}_3 = u_3$  and  $\mathcal{H}_{3,3} = u_4 + \frac{N-2}{2N} u_2^2$ . As we have seen, in order to recover the Seiberg-Witten solution, it is better to use the rescaled variables  $(\Lambda, \hat{T}_{n \geq 2})$ . It is straightforward to reexpress all the formulae in (3.8) as derivatives of  $\mathcal{F}$  with respect to  $\alpha^I, \hat{T}_n$  and  $\Lambda$ . Most of the factors  $T_1$  can be used to promote  $u_k$  to  $\hat{u}_k$  or, rather, to the homogeneous combinations thereof:

$$\hat{\mathcal{H}}_{m+1,n+1} = T_1^{m+n} \mathcal{H}_{m+1,n+1} \quad \Rightarrow \quad \hat{\mathcal{H}}_{m+1} = T_1^{m+1} \mathcal{H}_{m+1} \quad (3.9)$$

(since  $\mathcal{H}_{m+1} = \mathcal{H}_{m+1,2}$ ). The remaining  $T_1$ 's are absorbed in making up  $\hat{a}^I \equiv T_1 a^I(u_k, 1) = a^I(\hat{u}_k, T_1)$ . Altogether we find

$$\begin{aligned} \frac{\partial \mathcal{F}}{\partial \log \Lambda} &= \frac{\beta}{2\pi i} \sum_{n,m \geq 1} m \hat{T}_m \hat{T}_n \hat{\mathcal{H}}_{m+1,n+1} , & \frac{\partial \mathcal{F}}{\partial \hat{T}_n} &= \frac{\beta}{2\pi i n} \sum_{m \geq 1} m \hat{T}_m \hat{\mathcal{H}}_{m+1,n+1} , \\ \frac{\partial^2 \mathcal{F}}{\partial \alpha^I \partial \log \Lambda} &= \frac{\beta}{2\pi i} \sum_{m \geq 1} \hat{T}_m \frac{\partial \hat{\mathcal{H}}_{m+1}}{\partial \hat{a}^I} , & \frac{\partial^2 \mathcal{F}}{\partial \alpha^I \partial \hat{T}_n} &= \frac{\beta}{2\pi i n} \frac{\partial \hat{\mathcal{H}}_{n+1}}{\partial \hat{a}^I} , \\ \frac{\partial^2 \mathcal{F}}{\partial (\log \Lambda)^2} &= -\frac{\beta^2}{2\pi i} \sum_{m,n \geq 1} \hat{T}_m \hat{T}_n \frac{\partial \hat{\mathcal{H}}_{m+1}}{\partial \hat{a}^I} \frac{\partial \hat{\mathcal{H}}_{n+1}}{\partial \hat{a}^J} \frac{1}{i\pi} \partial_{\tau_{IJ}} \log \Theta_E(0|\tau) , & (3.10) \\ \frac{\partial^2 \mathcal{F}}{\partial \log \Lambda \partial \hat{T}_n} &= -\frac{\beta^2}{2\pi i n} \frac{\partial \hat{\mathcal{H}}_{n+1}}{\partial \hat{a}^I} \sum_{m \geq 1} \hat{T}_m \frac{\partial \hat{\mathcal{H}}_{m+1}}{\partial \hat{a}^J} \frac{1}{i\pi} \partial_{\tau_{ij}} \log \Theta_E(0|\tau) , \\ \frac{\partial^2 \mathcal{F}}{\partial \hat{T}_m \partial \hat{T}_n} &= -\frac{\beta}{2\pi i} \left( \hat{\mathcal{H}}_{m+1,n+1} + \frac{\beta}{mn} \frac{\partial \hat{\mathcal{H}}_{m+1}}{\partial \hat{a}^I} \frac{\partial \hat{\mathcal{H}}_{n+1}}{\partial \hat{a}^J} \frac{1}{i\pi} \partial_{\tau_{IJ}} \log \Theta_E(0|\tau) \right) , \end{aligned}$$

with  $m, n \geq 2$ . In these expressions  $\hat{T}_1 = 1$ . The restriction to the submanifold  $\hat{T}_2 = \hat{T}_3 = \dots = 0$  yields formulae which are expressed in terms of the original Seiberg-Witten data. Notice that in this subspace  $\alpha^I(u_k, T_1, \hat{T}_{n>1} = 0) = \hat{a}^I$ , hence

$$\frac{\partial \mathcal{F}}{\partial \log \Lambda} = \frac{\beta}{2\pi i} \hat{\mathcal{H}}_2 , \quad \frac{\partial \mathcal{F}}{\partial \hat{T}_n} = \frac{\beta}{2\pi i n} \hat{\mathcal{H}}_{n+1} ,$$

$$\begin{aligned}
\frac{\partial^2 \mathcal{F}}{\partial \hat{a}^I \partial \log \Lambda} &= \frac{\beta}{2\pi i} \frac{\partial \hat{\mathcal{H}}_2}{\partial \hat{a}^I} , & \frac{\partial^2 \mathcal{F}}{\partial \hat{a}^I \partial \hat{T}_n} &= \frac{\beta}{2\pi i n} \frac{\partial \hat{\mathcal{H}}_{n+1}}{\partial \hat{a}^I} , \\
\frac{\partial^2 \mathcal{F}}{\partial (\log \Lambda)^2} &= -\frac{\beta^2}{2\pi i} \frac{\partial \hat{\mathcal{H}}_2}{\partial \hat{a}^I} \frac{\partial \hat{\mathcal{H}}_2}{\partial \hat{a}^J} \frac{1}{i\pi} \partial_{\tau_{IJ}} \log \Theta_E(0|\tau) , \\
\frac{\partial^2 \mathcal{F}}{\partial \log \Lambda \partial \hat{T}_n} &= -\frac{\beta^2}{2\pi i n} \frac{\partial \hat{\mathcal{H}}_2}{\partial \hat{a}^I} \frac{\partial \hat{\mathcal{H}}_{n+1}}{\partial \hat{a}^J} \frac{1}{i\pi} \partial_{\tau_{ij}} \log \Theta_E(0|\tau) , \\
\frac{\partial^2 \mathcal{F}}{\partial \hat{T}_m \partial \hat{T}_n} &= -\frac{\beta}{2\pi i} \left( \hat{\mathcal{H}}_{m+1, n+1} + \frac{\beta}{mn} \frac{\partial \hat{\mathcal{H}}_{m+1}}{\partial \hat{a}^I} \frac{\partial \hat{\mathcal{H}}_{n+1}}{\partial \hat{a}^J} \frac{1}{i\pi} \partial_{\tau_{IJ}} \log \Theta_E(0|\tau) \right) ,
\end{aligned} \tag{3.11}$$

with  $m, n \geq 2$ . We will always work with the coordinates  $\Lambda$  and  $\hat{T}_n$  defined in (3.5) and, therefore, hats will be omitted everywhere.

The first derivatives of the prepotential with respect to the slow times can be interpreted as generalized RG equations. These equations show that, indeed, the slow times  $T_n$  are dual to the Casimirs  $\mathcal{H}_{n+1}$ , at least in the submanifold  $T_{n \geq 2} = 0$ . One can also work out the transformation properties under the symplectic group of the second derivatives of the prepotential. To do this, notice that the quantities  $\mathcal{H}_{n,m}$  are combinations of gauge invariant VEVs, therefore they are invariant under duality. The slow times are deformation parameters of the theory, and they are also invariant. The symplectic transformation of the periods  $\alpha^I$ ,  $\alpha_{D,I}$  are the usual ones for the variables  $a^I$ ,  $a_{D,I}$  of the Seiberg-Witten theory, and one derives the transformation law for the prepotential obtained in <sup>27), 28)</sup>

$$\mathcal{F}^T = \mathcal{F} + \frac{1}{2} \alpha^I (D^T B)_{ij} \alpha^j + \frac{1}{2} \alpha_{D,I} (C^T A)^{IJ} \alpha_{D,J} + \alpha^I (B^T C)_I^J \alpha_{D,J}. \tag{3.12}$$

One then finds <sup>18)</sup>,

$$\frac{\partial^2 \mathcal{F}}{\partial T_n \partial T_m} \rightarrow \frac{\partial^2 \mathcal{F}}{\partial T_n \partial T_m} - \frac{\partial^2 \mathcal{F}}{\partial T_n \partial \alpha^I} [(C\tau + D)^{-1}]^I_J C^{JK} \frac{\partial^2 \mathcal{F}}{\partial T_m \partial \alpha^K}. \tag{3.13}$$

This transformation law can also be checked on the explicit expressions in (3.10), by using the transformation law of the theta function under a symplectic transformation:

$$\Theta[\vec{\alpha}, \vec{\beta}](\vec{\xi}|\tau) \rightarrow e^{i\phi} (\det(C\tau + D))^{1/2} \exp \left[ \pi i \xi^t (C\tau + D)^{-1} C \xi \right] \Theta[\vec{\alpha}, \vec{\beta}](\vec{\xi}|\tau), \tag{3.14}$$

where  $e^{i\phi}$  is an eighth root of unity. The transformation law (3.13) is the generalization of (2.27) that we were looking for. We would be tempted to identify the contact term  $T_{k+1, l+1}$  with the second derivative  $\partial^2 \mathcal{F} / \partial T_k \partial T_l$ , up to a normalization factor. There are, however, two subtleties one has to take into account in order to make this identification. The first one is that, in the above model, the Casimirs are not  $u_k$ , but the homogeneous combinations  $\mathcal{H}_k$ . Therefore, to identify the contact terms with the second derivatives of the prepotential, we have to define the operators  $I_k(S)$  starting with  $\mathcal{H}_k$  instead of the  $u_k$  in the descent procedure. This is simply a redefinition of which gauge-invariant operators we take as a basis to construct the observables. The second subtlety is that the contact terms are not only determined by their transformation properties. As we discussed in the previous section,

there is the extra constraint that  $T_{k,l}$  vanishes semiclassically. Let's then analyze the semiclassical behavior of  $\partial^2 \mathcal{F} / \partial T_k \partial T_l$ . If we use the semiclassical expression for the effective gauge coupling  $\tau_{IJ}$ , we find that:

$$\partial_{\tau_{IJ}} \log \Theta_E(0|\tau) \sim \mathcal{O}\left(\frac{\Lambda^{2N}}{Z_{\vec{\alpha}_+}^{2N}}\right), \quad (3.15)$$

where  $Z_{\vec{\alpha}_+} = \vec{a} \cdot \vec{\alpha}_+$ ,  $\vec{a} = \sum_I a^I \vec{\alpha}_I$ , and  $\vec{\alpha}_+$  are the positive roots. Therefore, the term in  $\partial^2 \mathcal{F} / \partial T_k \partial T_l$  involving this derivative will vanish in the semiclassical region  $(\Lambda / Z_{\vec{\alpha}_+}) \rightarrow 0$ , for  $k, l = 1, \dots, N-1$  (we take  $T_1 = \Lambda$  here). On the other hand,  $\mathcal{H}_{n+1, m+1}$  does not vanish. Therefore,  $\partial^2 \mathcal{F} / \partial T_k \partial T_l$  does not have the required behavior for  $k, l = 2, \dots, N-1$ . This can be cured by a redefinition of the prepotential. Consider the “reduced” prepotential

$$\mathcal{F}^{\text{red}} = \mathcal{F} + \frac{\beta}{4\pi i} \sum_{n, m \geq 2} T_n T_m \mathcal{H}_{n+1, m+1}. \quad (3.16)$$

The second derivatives of this prepotential with respect to the slow times are, in the Seiberg-Witten submanifold  $T_{n \geq 2} = 0$ ,

$$\left( \frac{\partial^2 \mathcal{F}^{\text{red}}}{\partial T_n \partial T_m} \right)_{T_{n \geq 2} = 0} = -\frac{\beta^2}{2\pi i n} \frac{\partial \hat{\mathcal{H}}_{n+1}}{\partial \hat{a}^I} \frac{\partial \hat{\mathcal{H}}_{m+1}}{\partial \hat{a}^J} \frac{1}{i\pi} \partial_{\tau_{IJ}} \log \Theta_E(0|\tau), \quad (3.17)$$

and the contact terms are given by

$$T_{k+1, l+1} = \frac{\pi i k l}{\beta^2} \left( \frac{\partial^2 \mathcal{F}^{\text{red}}}{\partial T_k \partial T_l} \right)_{T_{n \geq 2} = 0}. \quad (3.18)$$

Notice that the piece that we subtracted in the prepotential has a nonzero classical limit, while (3.17) is a quantum correction. In other words, the classical limit of the original prepotential  $\mathcal{F}$  includes deformations which are quadratic in the slow times. This is not what we expect in the problem of evaluating the generating function (2.3). In this generating function, all the deformations are linear in the sources, and the quadratic terms in these sources that appear in the effective theory (the contact terms) are only due to quantum effects.

To summarize, we have seen that the prepotential defined in terms of the Whitham hierarchy is the appropriate object to understand the contact terms of the twisted theory. However, the above remarks suggest that there should be a way to improve the theory of the prepotential, in such a way that one obtains the reduced version introduced in (3.16), and that the slow times  $T_n$  can be associated to any homogeneous combination of the Casimirs of degree  $n+1$ .

#### §4. Blowup formula

In this section, we will discuss a further connection between integrable systems and the Coulomb integral, which appears when one considers the blowup formula. There are three reasons why such a formula is interesting: first of all, as shown in <sup>15)</sup>,

the behavior under blowup can be used to derive the contact terms. Second, as shown in <sup>13)</sup> and further discussed in <sup>16)</sup>, the blowup formula involves in a direct way the tau function of the periodic Toda lattice which underlies the Seiberg-Witten curve. Finally, using the blowup formula one can fix the characteristic  $E$  which appears in (3.8) <sup>18)</sup>. Of course, this formula is also very important in the context of Donaldson theory, as it makes possible to relate the Donaldson invariants of a manifold and its blowup <sup>\*</sup>). This formula was obtained using the  $u$ -plane integral in <sup>11)</sup>, for the case of  $SU(2)$ , and generalized to the higher rank case in <sup>13)</sup>.

Suppose that we have a four-manifold  $X$ , and we consider the blownup manifold at a point  $p$ ,  $\hat{X} = \text{Bl}_p(X)$ . Under this operation, the homology changes as follows <sup>30)</sup>:

$$H_2(X) \rightarrow H_2(\hat{X}) = H_2(X) \oplus \mathbf{Z} \cdot B, \quad (4.1)$$

where  $B$ , the class of the exceptional divisor, satisfies  $B^2 = -1$ . As  $b_2^+(\hat{X}) = b_2^+(X)$ , it makes sense to ask how the integrand of  $Z_{\text{Coulomb}}$  changes under the blowup. First of all, the blownup manifold  $\hat{X}$  has an extra two-homology class, therefore there are extra operators  $I_k(B)$  that must be included in the generating function. We will then write  $\hat{S} = S + tB$ . We want to compute then:

$$\left\langle \exp \left[ \sum_k (f_k I_k(S) + t_k I_k(B) + p_k \mathcal{O}_k) \right] \right\rangle_{\hat{X}}, \quad (4.2)$$

in terms of correlation functions of the twisted theory on  $X$ . In this equation,  $t_k \equiv f_k t$ . The first thing to do is to analyze the change in the integrand under blowup. This is easy to obtain if we also assume that the metric is such that  $(B, \omega) = 0$ . In this case, the lattice sum changes as follows:

$$\Psi_{\hat{X}} = \left( \sum_{n^I} \exp \left[ \pi i \tau_{IJ} n^I n^J + i \sum_k t_k \frac{\partial u_k}{\partial a^I} n^I - i\pi \sum_I n^I \right] \right) \Psi_X, \quad (4.3)$$

and we see that

$$\Psi_{\hat{X}} = \Theta[\vec{\Delta}, 0](\vec{\xi}|\tau) \Psi_X, \quad (4.4)$$

where  $\Theta[\vec{\Delta}, 0](\vec{\xi}|\tau)$  is a theta function with

$$\xi_I = \sum_k \frac{t_k}{2\pi} \frac{\partial u_k}{\partial a^I}, \quad \vec{\Delta} = (1/2, \dots, 1/2). \quad (4.5)$$

Notice that we have extracted from the Siegel-Narain theta function a standard theta function on the hyperelliptic curve (2.5). The above expression is valid in the electric frame, and the characteristic of the theta function is inherited from the term  $(\vec{\rho} \cdot \vec{\lambda}, w_2(X))$  in (2.16).

Let's now analyze the measure in the integrand. As  $\chi(\hat{X}) = \chi(X) + 1$  and  $\sigma(\hat{X}) = \sigma(X) - 1$ , the measure picks an extra factor

$$\left( \det \frac{\partial u_J}{\partial a^I} \right)^{1/2} \Delta_A^{-1/8} = \frac{1}{\Theta[\vec{\Delta}, 0](0|\tau)}, \quad (4.6)$$

---

<sup>\*</sup>) It is interesting to notice that the derivation of this formula in <sup>29)</sup> was one of the first hints of a relation between Donaldson theory and elliptic curves.

as a consequence of Thomae formulae. Putting all these factors together, we see that the blowup factor in the integrand is given (up to a constant) by:

$$\tau(t_k|u_k) = e^{-\sum t_k t_l T_{k,l}} \frac{\Theta[\vec{\Delta}, 0](\vec{\xi}|\tau)}{\Theta[\vec{\Delta}, 0](0|\tau)}. \quad (4.7)$$

As (4.7) is an extra factor in the integrand of  $Z_{\text{Coulomb}}$ , it follows by our arguments above that it must be invariant under duality transformations. Using (3.14), one can easily prove that the duality invariance of the blowup factor fixes the duality transformation of the contact terms, and one finds (2.24) again. The reason for the notation in (4.7) is that the blowup factor is essentially the tau function of the Toda hierarchy (see, for example, <sup>31</sup>). Notice that, in this identification, the coupling constants  $t_k$  are interpreted as the *fast* times of the hierarchy. Recall that there are two points of contact between integrable systems and the Seiberg-Witten solution: one is the fact that the hyperelliptic curves are spectral curves of periodic Toda chains (for pure Yang-Mills), and the other is that the prepotential is the logarithm of the *quasiclassical* tau function. Now, we see that the tau function itself appears in the twisted version of the theory as a blowup factor. This also gives a very direct link between Donaldson theory and its generalizations, and the hyperelliptic curve encoding the Seiberg-Witten solution.

In principle, the blowup factor depends on the moduli of the hyperelliptic curve in a complicated way. However, it turns out that the blowup factor only depends on the fast times  $t_k$  and on the Casimirs, as we have indicated with our notation. More precisely, let  $\vec{n} = (n_2, \dots, n_N)$  be a vector of nonnegative integers, with  $|\vec{n}| = n_2 + \dots + n_N$ . Then, there are *polynomials*  $\mathcal{B}_{\vec{n}}(u_2, \dots, u_N)$  in the Casimirs such that:

$$\tau(t_k|u_k) = \sum_{|\vec{n}| \geq 0} t_2^{n_2} \cdots t_N^{n_N} \mathcal{B}_{\vec{n}}(u_2, \dots, u_N). \quad (4.8)$$

The physical reason for this, as explained in <sup>11</sup>), is that one can interpret the blowup as a local “defect” created by an analog of the puncture operator. The effect of this puncture associated to the exceptional divisor should be represented by an infinite number of local observables. The ring of local observables is in fact generated by  $u_2, \dots, u_N$ , and an identity like (4.8) should be valid. One can indeed prove (4.8) by using duality invariance,  $R$ -charge arguments and regularity <sup>13</sup>). A consequence of (4.8) is the following relation between the generating functions:

$$\left\langle \exp \left[ \sum_k (f_k I_k(S) + t_k I_k(B) + p_k \mathcal{O}_k) \right] \right\rangle_{\tilde{X}} = \left\langle \exp \left[ \sum_k (f_k I_k(S) + p_k \mathcal{O}_k) \right] \tau(t_k | \mathcal{O}_k) \right\rangle_X. \quad (4.9)$$

Notice that the blowup factor  $\tau(t_k|u_k)$  does not depend on the manifold  $X$ . We have then proved that the blowup changes the generating function by a universal factor which can be expressed in terms of the zero observables.

As a corollary of (4.8), we can derive an explicit expression for the contact terms. The fact that the behavior under blowup can be used to obtain the contact terms was first remarked in <sup>15</sup>). To obtain the expression, we simply expand (4.7) to second

order in  $t_k$ . The first derivative of the theta function is zero due to the choice of characteristic. We then find:

$$\tau(t_k|u_k) = 1 - \sum_{k,l} \left( T_{k,l} + \frac{1}{2\pi i} \partial_{\tau_{IJ}} \log \Theta[\vec{\Delta}, \vec{0}](0|\tau) \frac{\partial u_k}{\partial a^I} \frac{\partial u_l}{\partial a^J} \right) t_k t_l + \dots, \quad (4.10)$$

Because of (4.8), this means that

$$T_{k,l} = -\frac{1}{2\pi i} \partial_{\tau_{IJ}} \log \Theta[\vec{\Delta}, \vec{0}](0|\tau) \frac{\partial u_k}{\partial a^I} \frac{\partial u_l}{\partial a^J} + \mathcal{B}_{\vec{n}_{k,l}}(u_2, \dots, u_N), \quad (4.11)$$

where  $\vec{n}_{k,l}$  are the vectors with  $|\vec{n}_{k,l}| = 2$ , corresponding to the quadratic terms in (4.8). The requirement that  $T_{k,l}$  vanishes semiclassically implies that  $\mathcal{B}_{\vec{n}_{k,l}}(u_2, \dots, u_N) = 0$ , and we finally find:

$$T_{k,l} = -\frac{1}{2\pi i} \partial_{\tau_{IJ}} \log \Theta[\vec{\Delta}, \vec{0}](0|\tau) \frac{\partial u_k}{\partial a^I} \frac{\partial u_l}{\partial a^J}, \quad (4.12)$$

which is the expression found in <sup>15)</sup>. If we compare now to (3.18), we see that they have the same structure. The only difference is due to the fact that the expression in (3.18) is valid if the descent operators are constructed with the operators  $\mathcal{H}_k$ . Here, instead, we are considering the operators that we defined in (2.2) starting from  $\mathcal{O}_k$ . An interesting consequence of this rederivation is that we can read the characteristic  $E$  in (3.8) from the expression (4.12):

$$E = (\vec{\Delta}, \vec{0}). \quad (4.13)$$

We will provide another check of this identification in the section 6.

To clarify our statement about the existence of the polynomials in (4.8), it is useful to consider in some detail the  $SU(2)$  case. The blowup factor is simply given by <sup>11)</sup>:

$$\tau(t|u) = e^{-t^2 T_{2,2}} \frac{\vartheta_4(\frac{t}{2\pi} \frac{du}{da} | \tau)}{\vartheta_4(0|\tau)}, \quad (4.14)$$

where  $\vartheta_4$  is the Jacobi theta form with characteristic  $[1/2, 0]$ , and we put  $u \equiv u_2$ . The quotient of these theta functions can be written in terms of a Weierstrass sigma function using the following identity <sup>32)</sup>:

$$\frac{\vartheta_4(z|\tau)}{\vartheta_4(0|\tau)} = e^{-\eta_2 \omega_2 z^2} \sigma_3(\omega_2 z), \quad (4.15)$$

where  $\omega_2$  is the  $a$ -period (*i.e.*,  $\tau = \omega_1/\omega_2$ ), and  $\eta_2 = \zeta(\omega_2/2)$ . Using the expression for the contact term in (2.29), the identity

$$\eta_2 = \frac{\pi^2}{6\omega_2} E_2(\tau), \quad (4.16)$$

and the fact that  $\omega_2 = (8\pi/\sqrt{2})(da/du)$ , one finds:

$$\tau(t|u) = e^{-t^2 \frac{u}{3}} \sigma_3\left(\frac{4t}{\sqrt{2}}\right) \quad (4.17)$$



This is precisely the expression found in<sup>29)</sup> in the context of Donaldson theory (for zero magnetic flux). The key fact is that the sigma functions  $\sigma_i(t)$  can be expanded around the origin, and the coefficients of the Taylor expansion are polynomials in the root  $e_i$ , and the functions  $g_2, g_3$  (in the Weierstrass description). These quantities depend only on  $u$ . From the Seiberg-Witten curve for the  $SU(2)$  pure Yang-Mills theory one finds in fact:

$$e_3 = -\frac{u}{12}, \quad g_2 = \frac{1}{4}\left(\frac{u^2}{3} - \frac{1}{4}\right), \quad g_3 = \frac{1}{48}\left(\frac{2u^3}{9} - \frac{u}{4}\right). \quad (4.18)$$

Therefore, the expansion of (4.17) is indeed of the form (4.8), and using that

$$\sigma_3(t) = 1 - e_3 t^2 + \mathcal{O}(t^4), \quad (4.19)$$

it follows that the quadratic term in  $t$  in (4.17) is zero, as expected from (4.11).

If we consider now the higher rank case, the above remarks on  $SU(2)$  suggest that the expansion in (4.8) should involve some kind of hyperelliptic generalization of the sigma functions. It is also quite possible that the interpretation in terms of the Toda-Whitham hierarchy gives a constructive way of computing these polynomials.

### §5. Soft Supersymmetry Breaking with Higher Casimir Operators

In this section, we will give a very rough overview of another use of Whitham hierarchies: soft supersymmetry breaking with spurion superfields.

The spurion formalism was introduced in<sup>33)</sup> and it is a very useful tool to break supersymmetry in an explicit way. The starting point of the spurion formalism is a coupling constant in a supersymmetric Lagrangian, call it  $m$ . This coupling constant can be promoted to a superfield,  $m \rightarrow M$ , and this will give another Lagrangian which will be supersymmetric as well. The superfield  $M$  is called a spurion superfield. Notice that the original Lagrangian is recovered by taking the VEV of the scalar component of  $M$  to be equal to  $m$ , and setting the rest of the fields to zero. To break supersymmetry, one gives a VEV to an *auxiliary* field in  $M$ . The resulting Lagrangian will be non-supersymmetric due to the extra terms generated in this way. An interesting example is the mass term for the  $\mathcal{N} = 2$  quark hypermultiplet, which in  $\mathcal{N} = 1$  superspace has the form:

$$\int d^2\theta m \tilde{Q} Q, \quad (5.1)$$

where  $\tilde{Q}, Q$  are the two  $\mathcal{N} = 1$  chiral superfields that correspond to the  $\mathcal{N} = 2$  quark hypermultiplet. If we promote  $m$  to an  $\mathcal{N} = 1$  chiral superfield:

$$m \rightarrow M = \phi_m + \sqrt{2}\theta\psi_m + \theta^2 F_m, \quad (5.2)$$

we obtain an  $\mathcal{N} = 1$  Lagrangian, and we recover the original one if we put  $\langle\phi_m\rangle = m$ . If we want to break supersymmetry, we take:

$$M = m + \theta^2 F_m. \quad (5.3)$$

This induces an extra term  $F_m \tilde{q} q$  (a mass term for the squark) which breaks supersymmetry down to  $\mathcal{N} = 0$ . The VEV of the auxiliary field,  $F_m$ , becomes a SUSY breaking parameter. Notice that one can also take  $m = 0$ ,  $F_m \neq 0$  to generate a mass term for the squarks while keeping the quarks massless. This is the usual procedure to decouple the squarks through soft supersymmetry breaking.

We can try to proceed in the same way with  $\mathcal{N} = 2$  gauge theories described by a prepotential. In the most general case, the prepotential is a function

$$\mathcal{F} = \mathcal{F}(\alpha^I, m_f, T_n), \quad (5.4)$$

where  $T_n$  are the Whitham slow times, and  $T_1 = \Lambda$ . In this Lagrangian, the variables  $\alpha^I$  correspond to  $\mathcal{N} = 2$  vector superfields, while the rest of the variables are couplings. Any of these couplings can be promoted to an  $\mathcal{N} = 2$  vector superfield, and then one breaks supersymmetry by giving VEVs to the auxiliary components. This strategy was followed in <sup>(24)-(26), (34)</sup> in the context of the original Seiberg-Witten theory, when  $T_n = 0$  for  $n \geq 2$ . The only coupling constants are then  $\Lambda$  and  $m_f$ . In the enlarged prepotential, we can construct a more general, nonsupersymmetric deformation of Seiberg-Witten theory, by promoting  $T_{n \geq 2}$  to spurion superfields while setting the VEVs of their scalar components to zero (for  $T_1$ , the scalar component has to be different from zero and equal to the quantum scale  $\Lambda$ ). It is convenient to define the couplings  $s_n$  as

$$s_1 = -i \log \Lambda, \quad s_n = -iT_n, \quad n = 2, \dots, r. \quad (5.5)$$

We then promote these couplings to  $\mathcal{N} = 2$  vector superfields  $S_n$ . Such a superfield can be written, in  $\mathcal{N} = 1$  language, as one  $\mathcal{N} = 1$  chiral superfield (that we will also denote by  $S_n$ ) and one  $\mathcal{N} = 1$  vector superfield  $V_n$ . We then have, in terms of  $\mathcal{N} = 1$  superfields,

$$S_1 = s_1 + \theta^2 F_1, \quad V_s \equiv V_1 = \frac{1}{2} D_1 \theta^2 \bar{\theta}^2, \quad (5.6)$$

$$S_n = \theta^2 F_n, \quad V_n = \frac{1}{2} D_n \theta^2 \bar{\theta}^2, \quad n \geq 2. \quad (5.7)$$

The Whitham times are then interpreted as supersymmetry breaking parameters, and supersymmetry is broken down to  $\mathcal{N} = 0$ . To obtain the explicit form of the soft breaking terms, we can expand the prepotential around  $S_{n \geq 2} = 0$ . The terms of order  $\mathcal{O}(S^3)$  in this expansion do not give any contribution to the Lagrangian, because they involve too many  $\theta$ 's and the integral in superspace will vanish. To analyze the softly broken theory, it is enough to consider the terms which are at most quadratic in the slow times, and the following expression is exact:

$$\mathcal{F}(\alpha^I, T_n) = \mathcal{F}(\alpha^I, T_1, 0) + \sum_{n \geq 2} \left( \frac{\partial \mathcal{F}}{\partial T_n} \right)_{T_{n \geq 2} = 0} T_n + \frac{1}{2} \sum_{n, m \geq 2} \left( \frac{\partial^2 \mathcal{F}}{\partial T_n \partial T_m} \right)_{T_{n \geq 2} = 0} T_n T_m. \quad (5.8)$$

The first term in this expansion is just the Seiberg-Witten prepotential, after setting  $T_1 = \Lambda$ . Notice that the prepotential defined in the Whitham framework is defined for the effective theory. Nevertheless, it is easy to see what is the classical prepotential by

going to the semiclassical region and switching off the quantum corrections. Again, it is more convenient to use the reduced prepotential (3.16). In this case, the second derivatives appearing in (5.8) vanish semiclassically, as we showed in (3.17). The first derivatives can be read from (3.11). They are written in terms of Casimir operators and therefore have a well-defined classical limit. We finally obtain the following expression for the reduced, classical prepotential:

$$\mathcal{F}_{\text{class}}^{\text{red}} = \frac{\beta}{2\pi} \sum_{n=1}^r \frac{1}{n} S_n \mathcal{H}_{n+1}, \quad (5.9)$$

where the spurions  $S_n$  are given in (5.6)–(5.7). The microscopic Lagrangian associated to (5.9) can be written as follows. First, one defines a generalized matrix of couplings as follows:

$$\tau_{ab}^{(\text{class})} = \frac{\partial^2 \mathcal{F}_{\text{class}}^{\text{red}}}{\partial \phi^a \partial \phi^b} = \tau \delta_{ab}, \quad (5.10)$$

$$\tau^{(\text{class})}{}_a{}^m = \frac{\partial^2 \mathcal{F}_{\text{class}}^{\text{red}}}{\partial \phi^a \partial S_m} = \frac{N}{\pi i m} \frac{\partial \mathcal{H}_{m+1}^{(\text{class})}}{\partial \phi^a} = \frac{N}{\pi i m} \text{tr}(\phi^m T_a) + \dots, \quad (5.11)$$

$$\tau_{(\text{class})}^{mn} = \frac{\partial^2 \mathcal{F}_{\text{class}}^{\text{red}}}{\partial S_m \partial S_n} = 0, \quad (5.12)$$

where the dots in eq.(5.11) denote the derivative with respect to  $\phi^a$  of lower order Casimir operators. The indices  $a, b, c, \dots$  belong to the adjoint representation of  $SU(N)$ , and are raised and lowered with the invariant metric, and  $n, m, \dots$  correspond to the variables  $s_n$ . The complex scalar field of the  $\mathcal{N} = 2$  vector multiplet is written as  $\phi = \sum_a \phi^a T_a$ , where  $T_a$  is a basis for the Lie algebra of  $SU(N)$ . Finally,  $\tau$  is the classical gauge coupling, and it is related to the spurion  $s_1 = -i \log \Lambda$  through the one loop formula  $\Lambda^\beta = e^{\pi i \tau}$ . We define now:

$$b^{(\text{class})} = \frac{1}{4\pi} \text{Im} \tau^{(\text{class})}. \quad (5.13)$$

We find

$$\begin{aligned} \mathcal{L} = \mathcal{L}_{N=2} - B_{(\text{class})}^{mn} \left( F_m F_n^* + \frac{1}{2} D_m D_n \right) + f_{bc}^e b^{(\text{class})}{}_a{}^m b^{(\text{class})-1}{}_{ae} D_m \phi^b \bar{\phi}^c \\ + \frac{1}{8\pi} \text{Im} \frac{\partial \tau^{(\text{class})}{}_b{}^m}{\partial \phi^a} \left[ (\psi^a \psi^b) F_m^* + (\lambda^a \lambda^b) F_m + i\sqrt{2}(\lambda^a \psi^b) D_m \right], \end{aligned} \quad (5.14)$$

where  $B_{(\text{class})}^{mn}$  is the classical value of the duality invariant quantity

$$B^{mn} = b_a{}^m b^{-1ab} b_b{}^n - b^{mn}, \quad (5.15)$$

and  $f_{bc}^a$  are the structure constants of the Lie algebra. In (5.14),  $\lambda$ ,  $\psi$  are the gluinos and  $\phi$  is the scalar component of the  $\mathcal{N} = 2$  vector superfield. We see that the spurion that corresponds to  $s_1$  gives mass to the gauginos of the  $\mathcal{N} = 2$  vector multiplet and to the imaginary part of the Higgs field  $\phi$ . The spurions corresponding

to higher Casimirs, on the other hand, give couplings between the Higgs field and the gauginos. Notice that the spurion superfields  $S_n$  have dimensions  $1-n$ , therefore the supersymmetry breaking parameters  $F_n$ ,  $D_n$  have dimension  $2-n$ . For  $n > 2$ , they will give nonrenormalizable (*i.e.* irrelevant) interactions in the microscopic Lagrangian. This does not mean that the resulting perturbations do not change the low-energy structure of the theory: the operators we are considering can be dangerously irrelevant operators, as in the related theory analyzed in <sup>36)</sup>, and in this case they will affect the infrared physics.

One of the advantages of this procedure to break supersymmetry is precisely that one can also write the exact low energy effective theory associated to (5.14), up to two derivatives. This is due to the fact that the dependence of the Lagrangian on the SUSY breaking parameters is controlled by the dependence of the prepotential on the spurions, and this is also given by the Seiberg-Witten solution. Notice that, in the low energy theory, the couplings (5.10)-(5.12) receive quantum corrections, and can be computed in terms of the hyperelliptic curve data. Using these couplings, one can write down an effective potential and study the vacuum structure of the theory. The effect of the nonsupersymmetric deformations associated to the higher order Casimirs in the vacuum structure of the theory has been explored in <sup>18)</sup>.

## §6. Instanton Corrections

In this section we will briefly review another use of the Whitham hierarchies: the computation of instanton corrections. For more details, see <sup>18)</sup>, <sup>35)</sup> and <sup>44)</sup>.

One of the main results of the Whitham approach to the theory of the prepotential have been the equations for the first and second derivatives of the prepotential in (3.8) and (3.10) derived in <sup>17)</sup>. In <sup>27)</sup> it was realized that the RG equation

$$\frac{\partial \mathcal{F}}{\partial \log \Lambda} = \frac{\beta}{4\pi i} u \quad (6.1)$$

is very useful to derive a recursion relation for the instanton contributions. In order to compute the instanton corrections, however, (6.1) is not sufficient and one needs additional input. This is usually provided by the Picard-Fuchs equations for the periods. The Picard-Fuchs equations are difficult to derive and solve when the rank of the gauge group is larger than one, although techniques from topological Landau-Ginzburg theories can make them more instrumental in order to obtain the one-instanton correction to the prepotential for the ADE series <sup>40)</sup>. It turns out that the equation for  $\partial^2 \mathcal{F} / \partial (\log \Lambda)^2$  in (3.11), together with (6.1), provides enough information to obtain the instanton expansion of the prepotential in the semiclassical region to any order, and we don't have to make use of the Picard-Fuchs equations. As we will see, the connection of  $SU(N)$ ,  $\mathcal{N} = 2$  super Yang-Mills theory with Toda-Whitham hierarchies embodies in a natural way a recursive procedure to compute all instanton corrections. The essential ingredient that makes this possible is

the relation of the derivatives of the prepotential with the theta function associated to the root lattice of the gauge group. To see how this works, consider the instanton expansion of the prepotential:

$$\mathcal{F} = \frac{1}{2N} \tau_0 \sum_{\vec{\alpha}_+} Z_{\vec{\alpha}_+}^2 + \frac{i}{4\pi} \sum_{\vec{\alpha}_+} Z_{\vec{\alpha}_+}^2 \log \frac{Z_{\vec{\alpha}_+}^2}{\Lambda^2} + \frac{1}{2\pi i} \sum_{k=1}^{\infty} \mathcal{F}_k(Z) \Lambda^{2Nk}. \quad (6.2)$$

In this expression,  $\sum_{\vec{\alpha}_+}$  denotes a sum over positive roots. The expansion is in powers of  $\Lambda^\beta$ , where  $\beta = 2N$  for  $SU(N)$ , and  $k$  is the instanton number. We then have

$$\frac{\partial^2 \mathcal{F}}{\partial(\log \Lambda)^2} = \frac{1}{2\pi i} \sum_{k=1}^{\infty} (2Nk)^2 \mathcal{F}_k(Z) \Lambda^{2Nk}, \quad (6.3)$$

which, according to (3.11), should be equal to

$$\frac{\partial^2 \mathcal{F}}{\partial(\log \Lambda)^2} = -\frac{\beta^2}{2\pi i} \frac{\partial \mathcal{H}_2}{\partial a^I} \frac{\partial \mathcal{H}_2}{\partial a^J} \frac{1}{i\pi} \partial_{\tau_{IJ}} \log \Theta_E(0|\tau). \quad (6.4)$$

The derivative of the quadratic Casimir has also an expansion that can be obtained from the RG equation and (6.2):

$$\frac{\partial \mathcal{H}_2}{\partial a^I} = \frac{2\pi i}{\beta} \frac{\partial^2 \mathcal{F}}{\partial a^I \partial \log \Lambda} = C_{IJ} a^J + \sum_{k=1}^{\infty} k \mathcal{F}_{k,I} \Lambda^{2Nk} \quad (6.5)$$

where  $C_{IJ}$  is the Cartan matrix,  $\mathcal{F}_{k,I} = \partial \mathcal{F}_k / \partial a^I$ , and we have taken into account that  $\frac{1}{2N} \sum_{\vec{\alpha}_+} Z_{\vec{\alpha}_+}^2 = \frac{1}{2} a^I C_{IJ} a^J$ . The couplings in the semiclassical region are obtained again from the expansion (6.2):

$$\tau_{IJ} = \frac{i}{2\pi} \sum_{\vec{\alpha}_+} \frac{\partial Z_{\vec{\alpha}_+}}{\partial a^I} \frac{\partial Z_{\vec{\alpha}_+}}{\partial a^J} \log \left( \frac{Z_{\vec{\alpha}_+}^2}{\Lambda^2} \right) + \frac{1}{2\pi i} \sum_{k=1}^{\infty} \mathcal{F}_{k,IJ} \Lambda^{2Nk}. \quad (6.6)$$

with  $\mathcal{F}_{k,IJ} = \frac{\partial^2 \mathcal{F}_k}{\partial a^I \partial a^J}$ . For convenience, in (6.6) a term  $\frac{i}{2\pi} \sum_{\vec{\alpha}_+} \frac{\partial Z_{\vec{\alpha}_+}}{\partial a^I} \frac{\partial Z_{\vec{\alpha}_+}}{\partial a^J} \left( \frac{2\pi i}{N} \tau_0 - 3 \right)$  has been set to zero by a suitable adjustment of the bare coupling  $2\pi i \tau_0 = 3N$ . If we set  $\vec{\alpha} = \sum_I n^I \vec{\alpha}_I$  and define

$$\vec{\alpha} \cdot \mathcal{F}_k'' \cdot \vec{\alpha} = \sum_{I,J} n^I \mathcal{F}_{k,IJ} n^J, \quad (6.7)$$

we see that the theta function  $\Theta_E$  in the semiclassical region can be written as

$$\Theta_E(0|\tau) = \sum_{r=0}^{\infty} \sum_{\vec{\alpha} \in \Delta_r} (-1)^{\vec{\rho} \cdot \vec{\alpha}} \prod_{\vec{\alpha}_+} Z_{\vec{\alpha}_+}^{-(\vec{\alpha} \cdot \vec{\alpha}_+)^2} \prod_{k=1}^{\infty} \left( \sum_{m=0}^{\infty} \frac{1}{2^m m!} (\vec{\alpha} \cdot \mathcal{F}_k'' \cdot \vec{\alpha})^m \Lambda^{2Nkm} \right) \Lambda^{2Nr}. \quad (6.8)$$

In the previous expression,  $\vec{\rho}$  is again the Weyl vector, and  $\Delta_r \subset \Delta$  is a subset of the root consisting of the lattice vectors  $\vec{\alpha}$  that fulfill the constraint  $\sum_{\vec{\alpha}_+} (\vec{\alpha} \cdot \vec{\alpha}_+)^2 = 2Nr$ . In particular  $\Delta_1$  is the root system, *i.e.* the simple roots together with their Weyl

reflections. In the above expression, we have used the characteristic (4.13), which is the appropriate one if we are working in the electric frame, as we should do in the semiclassical region. If we insert the expressions (6.5) and (6.8) into (6.4), and we equate the coefficients of the different powers of  $\Lambda$ , we find a set of recursive equations for  $\mathcal{F}_k$  which make possible to compute all the instanton coefficients by starting from the perturbative contribution to the prepotential. For example, one finds for the one-instanton correction:

$$\mathcal{F}_1 = - \sum_{\vec{\alpha} \in \Delta_1} Z_{\vec{\alpha}}^2 (-1)^{\vec{\rho} \cdot \vec{\alpha}} \prod_{\vec{\alpha}^+} Z_{\vec{\alpha}^+}^{-(\vec{\alpha} \cdot \vec{\alpha}^+)^2}, \quad (6.9)$$

and for the two-instanton contribution:

$$\begin{aligned} \mathcal{F}_2 = & -\frac{1}{4} \left( \sum_{\vec{\alpha} \in \Delta_1} (-1)^{\vec{\rho} \cdot \vec{\alpha}} \prod_{\vec{\alpha}^+} Z_{\vec{\alpha}^+}^{-(\vec{\alpha} \cdot \vec{\alpha}^+)^2} \left[ \mathcal{F}_1 + 2(\vec{\alpha} \cdot \mathcal{F}'_k) Z_{\vec{\alpha}} + \frac{1}{2}(\vec{\alpha} \cdot \mathcal{F}''_1 \cdot \vec{\alpha}) Z_{\vec{\alpha}}^2 \right] \right. \\ & \left. + \sum_{\vec{\beta} \in \Delta_2} Z_{\vec{\beta}}^2 (-1)^{\vec{\rho} \cdot \vec{\beta}} \prod_{\vec{\alpha}^+} Z_{\vec{\alpha}^+}^{-(\vec{\beta} \cdot \vec{\alpha}^+)^2} \right), \end{aligned} \quad (6.10)$$

where  $\vec{\alpha} \cdot \mathcal{F}'_k = \sum_I n^I \mathcal{F}_{k,I}$ . The above expression makes patent the recursive character of the procedure. We then see that, with this method, one can find the instanton corrections to the prepotential in a very simple way. The explicit expressions (6.9)–(6.10) are in full agreement with the results of<sup>(37), (39), (41)</sup> (see<sup>(18)</sup> for a detailed presentation). This agreement gives a further check of the RG equation (3.11) and of the choice of the characteristic (4.13).

These results on the instanton corrections can be extended in many ways. The equation (3.11) can be also used to study the strong coupling regime near the points of maximal singularity<sup>(42)</sup>. Moreover, one can analyze with this technique all the classical groups with or without matter content to find general expressions for the instanton corrections<sup>(44)</sup>.

## §7. Conclusions and Outlook

We have seen that the approach to the  $\mathcal{N} = 2$  prepotential based on the theory of Whitham hierarchies gives a very useful framework to understand deformations of  $\mathcal{N} = 2$  supersymmetric gauge theories. These deformations arise in different physical contexts. We have explored in detail the structure of the contact terms in the twisted version of  $\mathcal{N} = 2$  theories, and also the nonsupersymmetric deformations associated to soft supersymmetry breaking. In both cases, the approach based on Whitham hierarchies provides the right conceptual framework and the technical tools to derive the precise form of the deformed theory. We have also seen that one can obtain the instanton expansion in the semiclassical region by using the equations for the first and second derivatives of the prepotential with respect to the quantum scale.

Nevertheless, there are many issues that should be further clarified. The framework of Whitham hierarchies should be generalized to other gauge groups and matter content. Some steps in this direction can be found in<sup>(43)–(45)</sup>. In the context of the

twisted theory, one would like to introduce slow times associated to any homogeneous combination of the Casimirs, and construct a prepotential depending on these slow times and with the right semiclassical behavior. In this sense, although the explicit construction given in <sup>17)</sup> has clarified the theory of the prepotential and its applications to Donaldson theory, one should be able to improve it along the lines that we have suggested. It would be also nice to have an *a priori* connection between the Whitham hierarchy approach to the prepotential, and the structure of the generating function for the twisted theory. We have followed a rather indirect approach, based on a set of constraints for the contact terms and the RG equations. The approach of <sup>15)</sup>, based on Hamiltonian deformations of the prepotential, should be useful in deriving this connection.

It would be also very interesting to find an explicit construction of the polynomials appearing in (4.8) for the higher rank case. As we have suggested, this construction will involve interesting generalizations of sigma functions for hyperelliptic curves. Another reason for addressing this issue is that the structure of the blowup formula gives a very direct connection between the Toda hierarchy underlying the Seiberg-Witten solution, and the generalizations of Donaldson theory. This relation is certainly intriguing and suggests that many structures that have been found in two-dimensional topological gravity and two-dimensional topological matter could be also relevant in four dimensions.

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